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Linear Theory of Water Waves on a Running Stream

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SUMMARY

We consider the motion of an inviscid, incompressible fluid with surface tension T, in an infinite channel of finite depth, when a pressure disturbance is imposed on the uniform stream. The explicit solution of the resulting initial value problem is presented. Also, possible steady state flows are discussed. In the cases when they exist, corresponding radiation conditions are found.

1. Introduction

We consider an incompressible, non-viscous fluid with a surface tension T, moving in a uniform channel of finite depth which extends to infinity laterally. Our object is to study both the steady and the unsteady waves created by a disturbance imposed on the surface of a running stream. The disturbances considered are assumed to be so small that we can treat the problem within the framework of linearized theory.

The imposition of a disturbance on the surface of the running stream leads to a non-homogeneous boundary condition. The corresponding homogeneous problem is known to have non-trivial steady state solutions if $U^2 < gh$, with U, the initial velocity and h, the undisturbed depth of the fluid. As a result, the non-homogeneous problem for steady states does not, in general, have a unique solution, when boundedness conditions alone are imposed at infinity. However, the imposition of appropriate radiation conditions at infinity does lead to the existence of uniquely determined steady state solution. These conditions, however, are not known a priori.

Several devices, both of physical as well as of mathematical nature have been used to pick out a unique solution of the steady state problem without the direct imposition of radiation conditions. Michell [1], for example, chooses the solution which represents the gravity waves downstream only. The neglect of the part upstream amounts to the superposition of an appropriate solution of the homogeneous problem. Such a procedure is justified as long as the physical requirement, that in a moving frame of reference, the group velocity of the gravity waves is less than the phase velocity of the wave, is satisfied. Another device which is mathematical in nature has been used by Rayleigh [2]. This consists in introducing a small fictitious "dissipative force" which, without changing the character of the motion, makes the solution determinate. This new factor is then made to vanish in the final result. Lamb [3] uses this technique to discuss some cases of wave propagation. It turns out, however, that the solution thus obtained is not complete. There is a curious feature of the results of Lamb. These solutions involve a parameter U^2/gh. In case its value is one, the steady state solution has a large amplitude thus casting doubt even on the feasibility of the steady state solution in the framework of linear theory.

The point of view adopted here is that these odd phenomena stem in part from the unnaturalness of the steady state formulations in Newtonian mechanics. A way out, as has been considered by Green [4] and Stoker [5, 6], is to treat the complete initial value problem. The steady state solution, if it exists, should result from the unsteady solution in the limit as the time tends to infinity. This procedure has been followed here. Indeed, the results show that the transients die out in the supercritical and subcritical cases i.e. when $U^2 \gtrless gh$. In the critical case when

$U^2 = gh$ there is no steady state solution in the linear theory. We then consider the radiation conditions at the two ends. In the supercritical case, the disturbance dies out both upstream and downstream. In the subcritical case, it dies out upstream but not, in general, downstream. It, however, depends inversely on the surface tension. The wave length at infinity is fixed by the velocity of the stream U, the undisturbed stream depth h, and the surface tension of the medium. These results agree with those of Stoker for the case in which the medium does not admit surface tension.

I wish to express my gratitude to Professor J. J. Stoker for suggesting this problem to me. Thanks are also due to him, as well as to Professor A. S. Peters, for the fruitful discussions I had towards the completion of this work.

2. Formulation of the Problem

We consider a two-dimensional flow of liquid in a uniform channel which extends to infinity laterally and has a horizontal bottom. The liquid is taken to be inviscid, incompressible with a constant density ρ and surface tension T. We take the axis of x in the undisturbed free surface and the axis of y vertically upward. It is assumed that the motion is irrotational so that a velocity potential exists.

The liquid, initially, moves with a uniform velocity U in the positive x direction until a pressure disturbance $\Pi(x, t)$, $-\infty < x < \infty$, is imposed on the free surface. The motion is considered a small perturbation so that a linear theory can be applied. Let Φ, ϕ be the complete and the disturbed flow so that $\Phi(x, y, t) = \phi(x, y, t) + Ux$.

The differential equation governing the motion is

$$\nabla^2 \Phi(x, y, t) = 0 \qquad y < 0, \quad -\infty < x < \infty .$$

Bernoulli's law has the following form at the free surface

$$\frac{\Pi}{\rho} + g\eta + \Phi_t + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2) + \frac{T}{\rho} \frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0 \text{ for } y = \eta(x, t)$$

where $\eta(x, t)$ is the elevation of the free surface. The kinematic free surface condition and the condition at the bottom of the channel are:

$$\eta_t + \Phi_x \eta_x - \Phi_y = 0 \text{ at } y = \eta$$

and

$$\Phi_y = 0 \qquad \text{at } y = -h .$$

We further require that for any finite time t, the Fourier transforms of η, Π, ϕ and their derivatives in x, exist. This condition, reasonable from physical point of view, means that the disturbances initiated in a given region require time to reach distant parts.

At the time $t = 0$ the following initial conditions are imposed:

$$\phi(x, y, 0) = \eta(x, 0) = \Pi(x, 0) = 0$$

and

$$\Pi(x, t) \equiv \Pi(x) \text{ for } t > 0 .$$

They correspond to the uniform flow as initial state, with the horizontal free surface and the subsequent application of a steady pressure disturbance for $t > 0$.

The above equations are linearized with the following results:

$$\nabla^2 \phi = 0 \tag{1}$$

$$\frac{\Pi}{\rho} + g\eta + \phi_t + U\phi_x + \frac{T}{\rho}\eta_{xx} = 0 \text{ at } y = 0 \tag{2}$$

$$\eta_t + U\eta_x = \phi_y \qquad \text{at } y = 0 \tag{3}$$

$$\phi_y = 0 \qquad \text{at } y = -h \tag{4}$$

From (2) and the initial conditions, it follows that $\phi(x, y, 0)=\phi_t(x, y, 0)=0$ at $y=0$.

The problem then is to solve (1) in the strip $-h<y<0$ under the conditions mentioned above.

3. The Unsteady Wave Solution

We shall solve the problem by using the Fourier transform technique. The equations (1) and (4) transform into

$$\bar{\phi}_{yy}-s^2\bar{\phi}=0$$
$$\bar{\phi}_y(s, -h, t)=0$$

where the "bar" refers to the Fourier transform $\bar{\phi}(s, y, t)$ of $\phi(x, y, t)$. Thus

$$\bar{\phi}(s, y, t)=A(s, t)\cosh s(y+h)\,. \tag{6}$$

Also eliminating $\bar{\eta}$ from the transforms of the equations (2) and (3), we get,

$$\bar{\phi}_{tt}+2is\bar{\phi}_t-s^2U^2\bar{\phi}-\bar{\phi}_y\left(s^2\frac{T}{\rho}-g\right)=-is\frac{\bar{\Pi}}{\rho}\,. \tag{7}$$

The two yield the following differential equation to determine $A(s, t)$:

$$A_{tt}+2isUA_t-A\left\{s^2U^2+s\left(s^2\frac{T}{\rho}-g\right)\tanh sh\right\}=-isU\frac{\bar{\Pi}}{\rho}\operatorname{sech} sh\,.$$

The initial conditions for $A(s, t)$ are obtained by making use of (5). These are $A(s, 0)=0$, $A_t(s, 0)=0$. It then follows

$$A(s, t)=\frac{isU\bar{\Pi}}{\cosh sh}\left[\frac{1}{f_+\cdot f_-}+\frac{1}{2(T/\rho)^{\frac{1}{2}}k}\left(\frac{e^{-itf_+}}{f_+}-\frac{e^{-itf_-}}{f_-}\right)\right]$$

where $f_\pm=sU\pm(T/\rho)^{\frac{1}{2}}k(s)$

$k=\{s(a^2-s^2)\tanh sh\}^{\frac{1}{2}}$

$a^2=\rho g/T$

Inverting the transformation, we have

$$\phi(x, y, t)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}A(s, t)\cosh s(y+h)e^{isx}ds \tag{9}$$

Provided, we hold the three terms in the above expression for A together, we have to discuss the convergence of the integral (9) only at the two ends. For large values of s, the integrand, in absolute value varies as $|\bar{\Pi}(s)|s^{-2}e^{ys}e^{ts^{3/2}}$. We require that $\bar{\Pi}(s)=g(s)e^{-s^2}$ where $g(s)$ is an analytic function bounded at the two infinities. The integral then will converge uniformally for any finite value of t. It may be remarked that this form of $\bar{\Pi}(s)$ is quite appropriate, from physical viewpoint, because the disturbances in a stream, generally, are of local nature. Having obtained $\phi(x, y, t)$ we can use (2) or (3) to obtain the elevation $\eta(x, t)$ at the free surface.

4. Existence of Steady Flow

Our main interest now is to study what may happen when time t, tends to infinity. It is clear from the expression for A that the two out of the three terms represent the transients. These must vanish in the limit, if the steady state is to exist.

To study this, we shall change the path of integration in the complex s plane so as to avoid the singularities of $A(s, t)$ and then consider each integral individually. To begin with, let

$$F(s)=s^2U^2-\frac{T}{\rho}k^2(s)$$

Then for

$\dfrac{U^2}{gh} > 1$ $F(s)$ has a zero of second order at $s=0$

$\dfrac{U^2}{gh} = 1$ $F(s)$ has a zero of fourth order at $s=0$

$\dfrac{U^2}{gh} < 1$ $F(s)$ has a zero of second order at $s=0$ and two simple zeros at $s=\pm\beta$ where $|\beta|<a$.

Also we can define the function $A(s,t)$ as an analytic function in a small domain enclosed by the dotted lines in the s-plane. (See fig. 1)

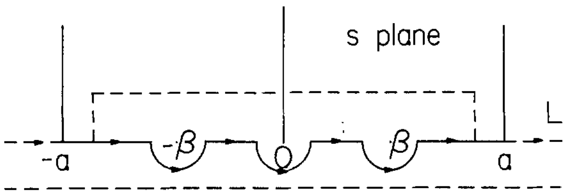

Figure 1

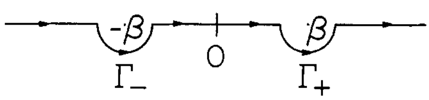

Figure 2

We then use Cauchy's theorem to change the path of integration of the integral (9) from the real line to the contour L as shown in the figure 1.

Case I $U^2 > gh$

Instead of considering the integral as such, we consider

$$\phi_x = \frac{i}{(2\pi)^{\frac{1}{2}}} \int_L sA(s,t)\cosh s(y+h)e^{isx}ds = I^{(x)} + I^{(t)}$$

where
$$I^{(x)} = \frac{-U}{2(2\pi\rho T)^{\frac{1}{2}}} \int_L \frac{\Pi s^2 \cosh s(y+h)}{f_+ \cdot f_-} e^{isx}ds$$

$$I^{(t)} = \frac{-U}{2(2\pi\rho T)^{\frac{1}{2}}} \int_L \frac{s^2\cosh s(y+h)}{\cosh sh\, k(s)} \left(\frac{\exp(-itf_+)}{f_+} - \frac{\exp(-itf_-)}{f_-} \right) e^{isx} ds$$

We wish to study the limit of $I^{(t)}$ as $t\to\infty$. The only zero of the denominator of the integrand, other than the branch points at $s=\pm a$, is a double zero at $s=0$ which is cancelled by the factor s^2 in the numerator. Hence we can change the contour back to the real line. We shall break the integral $I^{(t)}$ into 5 parts

$$I^{(t)} = I_{\infty}^{+(t)} + I_{\infty}^{-(t)} + I_a + I_{-\infty}^{+(t)} + I_{-\infty}^{-(t)}$$

where
$$I_{\infty}^{+(t)} = \frac{-U}{2(2\pi\rho T)^{\frac{1}{2}}} \int_a^{\infty} \frac{s^2\cosh s(y+h)}{\cosh sh\, k(s)} \frac{\exp(-itf_+)e^{isx}}{f_+} ds$$

and so on for the other integrals.

Let $s=a+t^3z^2$, then:

$$I_\infty^{+(t)} = \frac{-U}{2(2\pi\rho T)^{\frac{1}{2}}} \exp\{i(xa-Uta)\}t^{\frac{3}{2}}\int_0^\infty \psi(z,t)\exp\{iz^2(t^3x-Ut)\}dz$$

where $$\psi(z,t) = \left[\frac{(a+t^3z^2)^2 g(a+t^3z^2)\cosh(a+t^3z^2)(y+h)}{\cosh h(a+t^3z^2)\{(a+t^3z^2)(2a+t^3z^2)\tanh h(a+t^3z^2)\}^{\frac{1}{2}}}\right]$$

$$\cdot\left[iU(a+t^3z^2) - \frac{T}{\rho}(a+t^3z^2)(2a+t^3z^2)t^3z^2\tanh h(a+t^3z^2)^{\frac{1}{2}}\right]^{-1}$$

$$\cdot\left[\exp\left\{-(a+t^3z^2)^2+t^{11/2}z\left(\frac{T}{\rho}(z^2+at^{-3})(z^2+2at^{-3})\tanh h(a+t^3z^2)^{\frac{1}{2}}\right)\right\}\right].$$

Denoting these expressions by ψ_1, ψ_2, ψ_3, we have

$$\psi(z,t)=\psi_1\cdot\psi_2\cdot\psi_3$$
$$=\psi_1(-i\psi_2''-\psi_2')\cdot\psi_3=\psi'+i\psi''$$

where $\psi'=-\psi_1\cdot\psi_2'\cdot\psi_3$ and $\psi''=-\psi_1\cdot\psi_2''\cdot\psi_3$.

We have therefore to consider the integrals,

$$\frac{U}{2(2\pi\rho T)^{\frac{1}{2}}}\exp\{i(xa-Uta)\}t^{\frac{3}{2}}\int_0^\infty(\psi'+i\psi'')\exp\{iz^2(xt^3-Ut^4)\}dz$$

$$=\frac{U}{2(2\pi\rho T)^{\frac{1}{2}}}\exp\{i(xa-Uta)\}t^{\frac{3}{2}}\left[\left(\int_0^{(T/\rho)^{1/2}}+\int_{(T/\rho)^{1/2}}^\infty\right)(\psi'+i\psi'')\exp\{iz^2(xt^3-Ut^4)\}dz\right]$$

$$=J_1+J_2\text{, say}$$

where $$J_1=\frac{U}{2(2\pi\rho T)^{\frac{1}{2}}}\exp\{i(xa-Ut\bar{a})\}t^{\frac{3}{2}}\left[\int_0^{(T/\rho)^{1/2}}\psi^j\exp iz^2(xt^3-Ut^4)dz\right]$$

where $j='$ or $''$.

The function ψ^j, being analytic will have a finite number of zeros and stationary points in the interval $[0,(T/\rho)^{\frac{1}{2}}]$. This implies that it is possible to divide this interval into a finite number of subintervals such that ψ^j is monotonic in each one of them and maintains the same sign. Let one such interval be (α,β). We shall prove that the contribution, I, to the integral J_1, from this interval vanishes as $t\to\infty$. There are four cases: In (α,β), ψ^j is

(i) positive and monotonically decreasing
(ii) positive and monotonically increasing
(iii) negative and monotonically increasing
(iv) negative and monotonically decreasing.

Case (i)

$$I=\frac{U}{2(2\pi\rho T)^{\frac{1}{2}}}\exp\{i(ax-Uta)t^{\frac{3}{2}}\psi^j(\alpha,t)\left[\mathrm{Re}\int_\alpha^{\xi_1}+i\,\mathrm{Im}\int_\alpha^{\xi_2}\right]\exp\{i(-Ut^4+xt^3)z^2\}dz$$

$$\alpha<\xi_\nu<\beta;\quad \nu=1,2$$

Also

$$\int_\alpha^{\xi_\nu}\exp\{i(xt^3-Ut^4)z^2\}dz=\frac{\exp\{i(xt^3-Ut^4)\xi_\nu^2}{2i\xi_\nu(xt^3-Ut^4)}-\frac{\exp\{i(xt^3-Ut^4)\}\alpha^2}{2i\alpha(xt^3-Ut^4)}$$
$$+O(xt^3-Ut^4)^{-2}\quad\text{for}\quad\alpha\neq0$$
$$=\pi^{\frac{1}{2}}\exp\left(-\frac{\pi i}{4}\right)\{4|xt^3-Ut^4|\}^{-\frac{1}{2}}+O\{(Ut^4-xt^3)^{-1}\}\quad\text{for}\quad\alpha=0$$

Also it can be easily verified that $\lim_{t\to\infty} \psi^j(\alpha, t)$ is finite. Hence $I\to 0$ as $t\to\infty$.

Case (ii) we write

$$-I = \frac{U}{2(2\pi\rho T)^{\frac{1}{2}}} \exp\{i(ax-Uta)\}\, t^{\frac{3}{2}}$$
$$\cdot\left[\int_\alpha^\beta (\psi^j(\beta)-\psi^j(z)) \exp\{i(xt^3-Ut^4)z^2\}\,dz - \psi^j(\beta)\int_\alpha^\beta \exp\{i(xt^3-Ut^4)z^2\}\,dz\right]$$

Each of the two integrals on the right is of the same form as the one discussed in case (i).

The cases (iii) and (iv) reduce to the cases (i) and (ii) respectively if we replace ψ^j by $-\psi^j$. Hence $I\to 0$ as $t\to\infty$. Since (α, β) was any arbitrary subinterval, we conclude $J\to 0$ as $t\to\infty$. We now study the integral J_2.

$$|J_2| \leqq \frac{Ut^{\frac{3}{2}}}{2(2\pi\rho T)^{\frac{1}{2}}} \int_{(T/\rho)^{1/2}}^{\infty} |\psi|\,dz$$

It can easily be seen that for large values of t

$$|\psi_1| \leqq t^3 z^2 (\rho/T)^{\frac{1}{2}} M \quad \text{where} \quad |g(z)| \leqq M$$
$$|\psi_2| \leqq 1$$
$$|\psi_3| \leqq \exp(-t^6 z^4 + (T/\rho)^{\frac{1}{2}} z^3 t^{11/2})$$

so that

$$|J_2| \leqq \frac{Ut^{\frac{9}{2}}}{(8\pi)^{\frac{1}{2}}} \frac{T^{\frac{1}{2}}}{\rho^{\frac{3}{2}}} \int_1^\infty w^2 \exp\{t^6 (T/\rho)^2 (w^3-w^4)\}\,dw$$

where $\quad z=(T/\rho)^{\frac{1}{2}} w.$

Integrating by parts, we get,

$$|J_2| \leqq \frac{U\rho^{\frac{1}{2}}}{(8\pi)^{\frac{1}{2}} T^{\frac{3}{2}}} \frac{1}{t^{\frac{3}{2}}} \lim_{W\to\infty}\left[1-4\int_1^W \exp\frac{t^6 (T/\rho)^2 (w^3-w^4)}{(3-4w)^2}\,dw\right]$$
$$= \frac{U\rho^{\frac{1}{2}}}{(8\pi)^{\frac{1}{2}} T^{\frac{3}{2}}} \frac{1}{t^{\frac{3}{2}}} \lim_{W\to\infty}\left[1-\mu\left(\frac{1}{3-4W}+1\right)\right]\to 0 \quad \text{as} \quad t\to\infty$$

Hence $I^{+(t)}\to 0$ as $t\to\infty$. Also since $\mathrm{Re}(-itf_-)$ is negative, $I_\infty^{-(t)}\to 0$ as $t\to\infty$. In the same way we can prove that the third and the fourth integrals tend to zero as $t\to\infty$.

Finally we consider the integral I_a. The major contribution to this integral arises from the end points and from the stationary points of $f_\pm(s)$. As $f_-(s)$ is an odd extension of $f_+(s)$, it suffices to consider the equation $f'_+(s)=0$. These roots are the points of intersection of the positive branch of the curve

$$y^2 = U\{s(a^2-s^2)\tanh sh\}$$

with the curve

$$y = -\tfrac{1}{2}(T/\rho)\{(a^2-3s^2)\tanh sh + h(sa^2-s^3)\,\mathrm{sech}^2\, sh\}$$

and are all simple (see fig. 3). Using the method of stationary phase, we find $I_a \simeq O(t^{-\frac{1}{2}})\to 0$ as

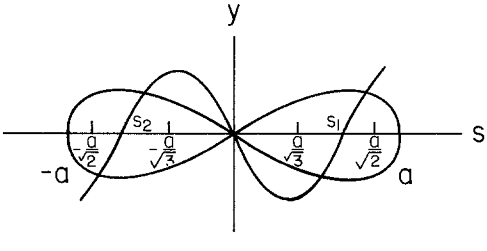

Figure 3

$t \to \infty$. We therefore conclude that the steady state solution exists.

Case II $U^2 = gh$

In this case it is more convenient to deal with the time derivative of the velocity potential:

$$\phi_t = \frac{U}{(8\pi\rho T)^{\frac{1}{2}}} \int_L \frac{s\bar{\Pi} \cosh s(y+h) e^{isx}}{k(s) \cosh sh} \{\exp(-itf_+) - \exp(-itf_-)\} ds$$

Aside from the integrable singularities at $s = \pm a$, the only singularities of the integrand are on the axis Im $s = 0$. The contour therefore can be changed back to the real axis. We shall again employ the method of stationary phase to get the first term in the asymptotic expansion of ϕ_t. The function $f'_+(s)$ has only simple zeros, while the function $f'_-(s)$ has a zero of second order at the origin and simple zeros elsewhere on the real line. The contribution from the neighborhood of the origin for large t, will dominate that from anywhere else. Hence

$$\phi_t \simeq A\bar{\Pi}(0) \frac{1}{|f_-(0)t|^{\frac{1}{3}}} \qquad A = \text{constant} \neq 0.$$

Also $\bar{\Pi}(0)$ in general is not zero. It follows

$$\phi \simeq O(t^{-\frac{1}{3}}),$$

i.e., the time dependent part of the velocity potential ϕ becomes infinite like $t^{\frac{2}{3}}$. This shows that the linear theory which assumes small disturbances fails to yield steady state solutions at the critical velocity $U = (gh)^{\frac{1}{2}}$. In this case one has to take recourse to the nonlinear theory.

Case III $U^2 < gh$.

We shall again show the time dependent part of ϕ_x tends to zero as $t \to \infty$. This case differs from the case I in that $f_+(s)$ and $f_-(s)$ each have an additional simple zero at $s = -\beta$ and $s = \beta$ respectively. We have therefore to consider the two integrals:

$$I_{\mp} = \frac{-U}{2(2\pi\rho T)^{\frac{1}{2}}} \int_{\Gamma_{\mp}} \frac{s^2 \cosh s(y+h) e^{isx}}{k(s) \cosh sh} \frac{e^{-itf_{\pm}}}{f_{\pm}} ds,$$

where $\Gamma_{\mp}$ are two small semicircles (excluding the end points) in the lower half of the s plane, centered at $s = \mp\beta$ respectively. (See figure 2). Here we have changed the contour L back to the real axis in the vicinity of the origin because the origin is no longer a singularity of the integrand. In order to see that $I \to 0$ as $t \to \infty$, we first notice that,

$$f'_-(\beta) = \frac{1}{2U} \{U^2(1 - 2h\beta \operatorname{cosech}(2h\beta) + 2\beta(T/\rho) \tanh h\beta\} > 0$$

Similarly $f'_+(-\beta) > 0$. Hence Re$\{-itf_+\}$ and Re$\{-itf_-\}$ are both negative on Γ_-, Γ_+. As a result $I \to 0$ as $t \to \infty$.

On the rest of the contour L, the conclusions of the case I hold. Hence the steady state again exists in this case.

5. Radiation Conditions for the Steady Flow

The radiation conditions for the resulting steady state:

$$\phi_x = \frac{-U}{(2\pi\rho T)^{\frac{1}{2}}} \int_L \frac{\bar{\Pi} s^2 \cosh s(y+h) e^{isx}}{s^2 U^2 - (T/\rho) s(a^2 - s^2) \tanh sh} ds$$

are obtained by considering the above integral in the limit $|x| \to \infty$.

In Case I i.e. when $U^2 > gh$, the integrand has no singularity on the real axis and so we deform the contour back to the real line. The resulting integral is readily seen to be absolutely convergent. Hence by the Rieman–Lebesgue Lemma, $\phi_x > 0$ as $x \to \infty$. It follows that the disturbance dies both upstream and the downstream. In the subcritical case, the situation turns out to be different. The contour is again deformed as shown in figure 2.

Let $x < 0$. On the semi-circular parts $\mathrm{Re}(isx) < 0$ and the rest of the function is bounded. Hence the contribution to ϕ_x from Γ_-, $\Gamma_+ \to 0$ as $x \to \infty$. On the straight parts we again invoke the Rieman-Lebesgue Lemma to conclude that the disturbance dies upstream. Let $x > 0$. The possible non-zero contributions result from the two semi-circles only. Arguing as above, it can be shown that the two semi-circles in the upper half plane centered at $s = \pm\beta$ yield vanishing contributions when $x \to \infty$. Hence we can evaluate the integrals for ϕ_x, for large x, over the complete circle. This can be done by the method of residue calculus. We have, then,

$$\phi_x \simeq i(2\pi)^{\frac{1}{2}} \beta^2 U \{\bar{\Pi}(\beta) e^{i\beta x} - \bar{\Pi}(-\beta) e^{-i\beta x}\} \frac{\cosh \beta(y+h)}{\cosh \beta h} H$$

where

$$\begin{aligned} H &= \lim_{s \to \beta} \frac{s-\beta}{s^2 U^2 - s(g - s^2(T/\rho)) \tanh sh} \\ &= \frac{1}{(g + \beta^2 T/\rho) \tanh h\beta - (g\beta - \beta^3 T/\rho) \operatorname{sech}^2 h\beta}, \quad |\beta| < (g/T)^{\frac{1}{2}} \end{aligned}$$

The disturbance downstream, therefore, does not, in general, die.

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